

Bijjective 1-cocycles and Classification of 3-dimensional Left-symmetric Algebras

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Abstract

Left-symmetric algebras have close relations with many important fields in mathematics and mathematical physics. Their classification is very complicated due to the nonassociativity. In this paper, we re-study the correspondence between left-symmetric algebras and the bijective 1-cocycles. Then a procedure is provided to classify left-symmetric algebras in terms of classification of equivalent classes of bijective 1-cocycles. As an example, the 3-dimensional complex left-symmetric algebras are classified.

Key Words Left-symmetric algebra; Lie algebra; 1-cocycle

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1 Introduction

Left-symmetric algebras (or under other names like Koszul-Vinberg algebras, quasi-associative algebras, pre-Lie algebras and so on) are a class of nonassociative algebras coming from the study of several topics in geometry and algebra, such as rooted tree algebras ([C]), convex homogenous cones ([V]), affine manifolds and affine structures on Lie groups ([Ko],[Ma]), deformation of associative algebras ([G]) and so on. They are Lie-admissible algebras (in the sense that the commutators define Lie algebra structures) whose left multiplication operators form a Lie algebra.

Furthermore, left-symmetric algebras are a kind of natural algebraic systems appearing in many fields in mathematics and mathematical physics. Perhaps this is one of the most attractive and interesting places. As it was pointed out in [CL], the left-symmetric algebra “deserves more attention than it has been given”. For example, left-symmetric algebras appear as an underlying structure of

those Lie algebras that possess a phase space, thus “they form a natural category from the point of view of classical and quantum mechanics” ([Ku1-2]); they are the underlying algebraic structures of vertex algebras ([BK]); there is a correspondence between left-symmetric algebras and complex product structures on Lie algebras ([AS]), which plays an important role in the theory of hypercomplex and hypersymplectic manifolds ([Bar]); left-symmetric algebras have close relations with certain integrable systems ([Bo],[LM]), classical and quantum Yang-Baxter equation ([DM],[ESS],[GS],[Ku3]), Poisson brackets and infinite-dimensional Lie algebras ([BN],[DN],[GD]), operads ([CL]), quantum field theory ([CK]) and so on (see [Bu3] and the references therein).

On the other hand, it is hard to study left-symmetric algebras. Due to the nonassociativity, there is neither a suitable representation theory nor a complete structure theory like other classical algebras such as associative algebras and Lie algebras. Even there exist simple transitive left-symmetric algebras which combine the simplicity and certain nilpotence ([H],[Bu1-2], or see the type $(D_{-1} - 10)$ in section 3). In fact, many fundamental problems have not been solved. Even the classification of complex left-symmetric algebras is only up to dimension 2 ([BM1], [Bu2]).

Therefore we hope to get more examples which can be regarded as a guide for further study. One of the ideas to get examples is to use some well-known structures to obtain some left-symmetric algebras (the so-called “realization” theory). We have already obtained some experiences. For example, a commutative associative algebra (A, \cdot) and its derivation D can define a Novikov algebra $(A, *)$ (which is a left-symmetric algebra with commutative right multiplication operators) as follows ([GD],[BM4-5]):

$$x * y = x \cdot Dy, \quad \forall x, y \in A. \quad (1.1)$$

An analogue of the above construction in the version of Lie algebras is related to the classical Yang-Baxter equation. In fact, a Lie algebra $(\mathcal{G}, [\cdot, \cdot])$ and a linear map $R : \mathcal{G} \rightarrow \mathcal{G}$ satisfying the (operator form) of classical Yang-Baxter equation ([S])

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]), \quad \forall x, y \in \mathcal{G} \quad (1.2)$$

can define a left-symmetric algebra $(\mathcal{G}, *)$ as follows ([BM6],[GS],[Me]):

$$x * y = [R(x), y], \quad \forall x, y \in \mathcal{G}. \quad (1.3)$$

Moreover, equation (1.3) also gives an algebraic interpretation of the so-called “left-symmetry”: in some sense, the “left-symmetry” can be interpreted as a Lie bracket “left-twisted” by a classical r -matrix. Furthermore, the above construction can be generalized to any representation (ρ, V) of a

Lie algebra \mathcal{G} as follows. Let $T : V \rightarrow \mathcal{G}$ be a linear map satisfying

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(v))u), \quad \forall u, v \in \mathcal{G}. \quad (1.4)$$

Such a map is called an \mathcal{O} -operator in [Ku3] which satisfies the (generalized) classical Yang-Baxter equation (in fact, it is a solution of classical Yang-Baxter equation on a larger Lie algebra ([Bai])). Then there exist left-symmetric algebra structures in both V and $T(V) \subset \mathcal{G}$ given by

$$u * v = \rho(T(u))v; \quad T(u) * T(v) = T(\rho(T(u))v), \quad \forall u, v \in V, \quad (1.5)$$

respectively. These relations are not only useful for the study of the left-symmetric algebras themselves such as giving more examples as above and illuminating some interesting properties, but also can provide those related topics with certain algebraic and geometric interpretations.

Note that if the \mathcal{O} -operator T appearing in equation (1.4) is invertible, then the operator T^{-1} is just a 1-cocycle associated to the representation (ρ, V) of \mathcal{G} . In fact, there is a closer relation between left-symmetric algebras and bijective 1-cocycles: there exists a compatible left-symmetric algebra structure on a Lie algebra \mathcal{G} if and only if \mathcal{G} has a bijective 1-cocycle. In this paper, we re-study the correspondence between left-symmetric algebras and bijective 1-cocycles. Although most of the results have been already known ([Ki1], [Me]), our discussion can provide a procedure to classify left-symmetric algebras using the representation theory of Lie algebras. It is a “linearization” method which avoids classifying the (non-linear) quadratic forms of structural constants. In particular, it is quite effective for the classification of complex left-symmetric algebras in low dimensions, such as in dimension 3.

The paper is organized as follows. In Section 2, we re-study the correspondence between the left-symmetric algebras and bijective 1-cocycles. In Section 3, we give the classification of 3-dimensional complex left-symmetric algebras.

Throughout this paper, without special saying, all algebras are of finite dimension and over an algebraically closed field of characteristic 0.

2 Left-symmetric algebras and bijective 1-cocycles

2.1 Preliminaries on left-symmetric algebras

Definition 2.1 Let A be a vector space over a field \mathbf{F} equipped with a bilinear product $(x, y) \rightarrow xy$. A is called a left-symmetric algebra if for any $x, y, z \in A$, the associator

$$(x, y, z) = (xy)z - x(yz) \quad (2.1)$$

is symmetric in x, y , that is,

$$(x, y, z) = (y, x, z), \text{ or equivalently } (xy)z - x(yz) = (yx)z - y(xz). \quad (2.2)$$

For a left-symmetric algebra A , the commutator

$$[x, y] = xy - yx, \quad (2.3)$$

defines a Lie algebra $\mathcal{G} = \mathcal{G}(A)$, which is called the sub-adjacent Lie algebra of A . For any $x, y \in A$, let L_x and R_x denote the left and right multiplication operator respectively, that is, $L_x(y) = xy$, $R_x(y) = yx$. Then the left-symmetry (2.2) is just

$$[L_x, L_y] = L_{[x, y]}, \quad \forall x, y \in A, \quad (2.4)$$

which means that $L : \mathcal{G}(A) \rightarrow gl(\mathcal{G}(A))$ with $x \rightarrow L_x$ gives a (regular) representation of the Lie algebra $\mathcal{G}(A)$.

Some subclasses of left-symmetric algebras are very important.

Definition 2.2 Let A be a left-symmetric algebra.

(1) If A has no ideals except itself and zero, then A is called simple. A is called semisimple if A is the direct sum of simple left-symmetric algebras.

(2) If for every $x \in A$, R_x is nilpotent, then A is said to be transitive or complete. The transitivity corresponds to the completeness of an affine manifold ([Ki1],[Me]). Moreover, the sub-adjacent Lie algebra of a transitive left-symmetric algebra is solvable.

(3) If for every $x, y \in A$, $R_x R_y = R_y R_x$, then A is called a Novikov algebra. It was introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in the formal variational calculus ([BN],[BM3-6],[GD],[O],[X],[Z]).

(4) If for every $x, y, z \in A$, the associator (x, y, z) is right-symmetric, that is, $(x, y, z) = (x, z, y)$, then A is said to be bi-symmetric. It is just the assosymmetric ring in the study of near associative algebras ([Kl],[BM2]).

2.2 Bijective 1-cocycles

Definition 2.3 Let \mathcal{G} be a Lie algebra and $f : \mathcal{G} \rightarrow gl(V)$ be a representation of \mathcal{G} . A 1-cocycle q associated to f is defined as a linear map from \mathcal{G} to V satisfying

$$q[x, y] = f(x)q(y) - f(y)q(x), \quad \forall x, y \in \mathcal{G}. \quad (2.5)$$

We denote it by (f, q) . In addition, if q is a linear isomorphism (thus $\dim V = \dim \mathcal{G}$), (f, q) is said to be bijective.

Let (f, q) be a bijective 1-cocycle, then it is easy to see that

$$x * y = q^{-1}(f(x)q(y)), \quad \forall x, y \in \mathcal{G}. \quad (2.6)$$

defines a left-symmetric algebra on \mathcal{G} ([Me]). Conversely, for a left-symmetric algebra \mathcal{G} , (L, id) is a bijective 1-cocycle of \mathcal{G} . Hence we have the following maps:

$$\Phi : \mathbf{A} = \{\text{bijective 1-cocycles}\} \longrightarrow \mathbf{B} = \{\text{left-symmetric algebras}\}$$

$$\Psi : \mathbf{B} = \{\text{left-symmetric algebras}\} \longrightarrow \mathbf{A} = \{\text{bijective 1-cocycles}\}$$

Definition 2.4 Let \mathcal{G} be a Lie algebra. Let (f_1, V_1) and (f_2, V_2) be two linear representations and q_1, q_2 be bijective 1-cocycles associated to f_1, f_2 respectively. (f_1, V_1) is isomorphic (\cong) to (f_2, V_2) if there exists a linear isomorphism $g : V_1 \rightarrow V_2$ such that $f_2 = gf_1g^{-1}$. We call them equivalent (\sim) if there exists an automorphism T of \mathcal{G} such that $(f_1T, V_1) \cong (f_2, V_2)$. (f_1, q_1) is isomorphic (\cong) to (f_2, q_2) if there exists a linear isomorphism $g : V_1 \rightarrow V_2$ such that $f_2 = gf_1g^{-1}$ and $q_2 = gq_1$. We call them equivalent (\sim) if there exists an automorphism T of \mathcal{G} such that $(f_1T, q_1T) \cong (f_2, q_2)$, that is, there exist a linear isomorphism $g : V_1 \rightarrow V_2$ and an automorphism $T : \mathcal{G} \rightarrow \mathcal{G}$ such that $f_2 = gf_1Tg^{-1}$ and $q_2 = gq_1T$.

On the other hand, recall that two left-symmetric algebras $(\mathcal{G}_1, *)$ and (\mathcal{G}_2, \cdot) are isomorphic (denoted by \cong) if there exists a linear isomorphism $F : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ such that $F(x * y) = F(x) \cdot F(y)$ for any $x, y \in \mathcal{G}_1$.

Theorem 2.1 The maps Φ and Ψ induce a bijection between the set of the isomorphism classes of bijective 1-cocycles of \mathcal{G} and the set of left-symmetric algebras on \mathcal{G} . Under this correspondence equivalent bijective 1-cocycles are mapped to isomorphic left-symmetric algebras and vice versa. That is,

$$\mathbf{A}/\cong \longleftrightarrow \mathbf{B}; \quad \mathbf{A}/\sim \longleftrightarrow \mathbf{B}/\cong. \quad (2.7)$$

Proof Let (f_1, q_1) and (f_2, q_2) be two isomorphic bijective 1-cocycles. Then there exists a linear isomorphism g such that $gf_1 = f_2g$ and $q_2 = gq_1$. We can know their corresponding left-symmetric algebras coincide since

$$x * y = q_1^{-1}(f_1(x)q_1(y)) = (q_2^{-1}g)[(g^{-1}f_2(x)g)(g^{-1}q_2)(y)] = q_2^{-1}(f_2(x)q_2(y)), \quad \forall x, y \in \mathcal{G}.$$

Therefore the map Φ is defined on the set of isomorphism classes of bijective 1-cocycles.

Let (f, q) be a bijective 1-cocycle, then $\Psi\Phi(f, q) \cong (f, q)$ by $g = q$. Conversely, from the definitions, we know that $\Phi\Psi(*) = *$, that is, $\Phi\Psi$ maps any left-symmetric algebra to itself. Hence the correspondence is proved.

Now we prove that equivalent bijective 1-cocycles correspond to isomorphic left-symmetric algebras. Let (f_1, q_1) and (f_2, q_2) be two equivalent bijective 1-cocycles. Then there exists a linear isomorphism there exist a linear isomorphism $g : V_1 \rightarrow V_2$ and an automorphism $T : \mathcal{G} \rightarrow \mathcal{G}$ such that $f_2 = gf_1Tg^{-1}$ and $q_2 = gq_1T$. Their corresponding left-symmetric algebra are given by

$$x *_1 y = q_1^{-1}(f_1(x)q_1(y)), \quad \forall x, y \in \mathcal{G}$$

and

$$x *_2 y = q_2^{-1}(f_2(x)q_2(y)) = (T^{-1}q_1^{-1}g^{-1})[(gf_1T(x)g^{-1})(gq_1T(y))], \quad \forall x, y \in \mathcal{G},$$

respectively. So we have

$$Tx *_1 Ty = T(x *_2 y). \quad \forall x, y \in \mathcal{G}.$$

Hence these two left-symmetric algebras are isomorphic by T . Conversely, let F be an (left-symmetric algebra) automorphism of \mathcal{G} . Obviously, F is also a Lie algebra automorphism of \mathcal{G} and (LF, F) is a bijective 1-cocycle of \mathcal{G} corresponding to the image of F . Then the bijective 1-cocycle (L, id) is equivalent to the bijective 1-cocycle (LF, F) by $g = id$. \square

Hence the classification of left-symmetric algebras in the sense of isomorphism is as the same as the classification of bijective 1-cocycles in the sense of equivalence.

Remark 1 The above correspondence is similar to the correspondence between left-symmetric algebras and étale affine representations given in [Bau].

Remark 2 As in the introduction, when (f, q) is a bijective 1-cocycle of a Lie algebra \mathcal{G} , then q^{-1} is an \mathcal{O} -operator associated to f , that is, q^{-1} satisfies the (generalized) classical Yang-Baxter equation. Moreover, equation (2.6) coincides with the latter part of equation (1.5) since

$$x * y = q^{-1}(u) * q^{-1}(v) = q^{-1}(f(q^{-1}(u))v) = q^{-1}(f(x)q(y)), \quad (2.8)$$

where $x = q^{-1}(u), y = q^{-1}(v)$.

2.3 Classification Problems

Due to the nonassociativity, it is very difficult to classify left-symmetric algebras. A natural way is to classify the structural constants, which has been used in dimension 2 ([BM1] or [Bu2]). However, it can not be extended to the higher dimensions since it involves the classification of quadratic forms of

structural constants, which is very complicated due to nonlinearity, even in dimension 3. Moreover, unlike associative algebras or Lie algebras, there is not a complete structure theory. For example, although there are several definitions of radicals ([Bu1-2],[H],[Me]), none is good enough. In fact, up to now, there are only 2-dimensional complex left-symmetric algebras and some special cases in higher dimensions (for example, transitive cases on nilpotent Lie algebras up to dimension 4 ([Ki1-2]) and bi-symmetric algebras ([BM2]) and Novikov algebras ([BM3]) up to dimension 3) have been classified.

From the relation between left-symmetric algebras and bijective 1-cocycles, we can solve this problem by classifying the equivalent bijective 1-cocycles. In fact, we can divide the classification into several steps:

Step 1: Classify Lie algebras. This has been done in certain low dimensions and some special cases ([J],[SW]).

Step 2: Let \mathcal{G} be a given Lie algebra with a basis $\{e_1, \dots, e_n\}$. Compute its automorphism group $\text{Aut}(\mathcal{G})$. For a representation $f : \mathcal{G} \rightarrow gl(V)$ ($\dim V = \dim \mathcal{G}$) with a basis $\{v_1, \dots, v_n\}$ of V , we can let $f(x) = (f_{ij}(x))$ for any $x \in \mathcal{G}$, where $f_{ij} : \mathcal{G} \rightarrow \mathbf{F}$ be linear functions. On the other hand, let $q : \mathcal{G} \rightarrow V$ be a 1-cocycle, then we can let $q(x) = \sum_{k=1}^n A_k(x)v_k$, where $A_k : \mathcal{G} \rightarrow \mathbf{F}$ are linear functions. The conditions of the representation f and the 1-cocycle q can give a series of equations for linear functions f_{ij} and A_k .

Step 3: Classify the linear functions f_{ij} under the sense of equivalence through the basis transformations of V and the basis transformation of \mathcal{G} which is compatible with the automorphism group of \mathcal{G} .

Step 4: For a given representation obtained in step 3, find all the corresponding bijective 1-cocycles (that is, the determinant of $(A_j(e_i))$ is non-zero).

Step 5: Classify those bijective 1-cocycles in step 4 and their corresponding left-symmetric algebras.

Although it seems that it is more complicated to classify bijective 1-cocycles than left-symmetric algebras themselves, in fact, there are certain advantages: every above step only involves linear equations, thus avoiding the classification of the nonlinearity of structural constants; the whole classification is like a kind of “variable separated” (in particular in step 3 and step 5). The whole process is like a kind of “linearization” of classifying structural constants of left-symmetric algebras. Moreover, this method can be extended to use the extensions of left-symmetric algebras ([Ki2]).

The above procedure will be quite effective to classify some left-symmetric algebras over the

complex number \mathbf{C} . As an example, we give the classification of 3-dimensional complex left-symmetric algebras in Section 3.

3 The classification of 3-dimensional complex left-symmetric algebras

It is well-known that there does not exist any left-symmetric algebra structure on a complex semisimple Lie algebra (cf. [Me]). Hence, over the complex field \mathbf{C} , besides 3-dimensional simple Lie algebra $sl(2, \mathbf{C})$, up to isomorphisms, there are the following (non-isomorphic) Lie algebras ([J]): (we only give the non-zero products)

(a) Abelian Lie algebra;

(b) Heisenberg Lie algebra $\mathcal{H} = \langle e_1, e_2, e_3 | [e_1, e_2] = e_3 \rangle$;

(c) $\mathcal{N} = \langle e_1, e_2, e_3 | [e_3, e_2] = e_2 \rangle$, which is a direct sum of 2-dimensional non-abelian Lie algebra and 1-dimensional center;

(d) $\mathcal{D}_l = \langle e_1, e_2, e_3 | [e_3, e_1] = e_1, [e_3, e_2] = le_2 \rangle$, $0 < |l| < 1$ or $l = e^{i\theta}$, $0 \leq \theta \leq \pi$;

(e) $\mathcal{E} = \langle e_1, e_2, e_3 | [e_3, e_1] = e_1, [e_3, e_2] = e_1 + e_2 \rangle$.

All of the above Lie algebras are solvable. Let \mathcal{G} be one of these algebras and $f : \mathcal{G} \rightarrow gl(V)$ be a representation. Thus, according to Lie Theorem ([J]), there exists a basis $\{v_1, v_2, v_3\}$ of V such that

$$f(x) = \begin{pmatrix} f_{11}(x) & 0 & 0 \\ f_{21}(x) & f_{22}(x) & 0 \\ f_{31}(x) & f_{32}(x) & f_{33}(x) \end{pmatrix}, \forall x \in \mathcal{G} \quad (3.1)$$

where $f_{11}, f_{22}, f_{33}, f_{21}, f_{32}, f_{31}$ are linear functions of \mathcal{G} . Let $q : \mathcal{G} \rightarrow V$ be 1-cocycle:

$$q(x) = A_1(x)v_1 + A_2(x)v_2 + A_3(x)v_3, \quad (3.2)$$

where A_1, A_2, A_3 are linear functions of \mathcal{G} . The matrix associated to q is defined as $C = (A_j(e_i)) = \begin{pmatrix} A_1(e_1) & A_2(e_1) & A_3(e_1) \\ A_2(e_1) & A_2(e_2) & A_3(e_3) \\ A_3(e_1) & A_3(e_2) & A_3(e_3) \end{pmatrix}$. q is bijective if and only if $\det C \neq 0$.

For a left-symmetric algebra \mathcal{G} , the (form) characteristic matrix is defined as

$$M = \begin{pmatrix} \sum_{k=1}^n a_{11}^k e_k & \cdots & \sum_{k=1}^n a_{1n}^k e_k \\ \cdots & \cdots & \cdots \\ \sum_{k=1}^n a_{n1}^k e_k & \cdots & \sum_{k=1}^n a_{nn}^k e_k \end{pmatrix}, \quad (3.3)$$

where $\{e_i\}$ is a basis of \mathcal{G} and $e_i e_j = \sum_{k=1}^n a_{ij}^k e_k$.

The left-symmetric algebras on abelian Lie algebras are commutative associative algebras which are classified in dimension 3 in [BM3]. In the next subsections, we give the classification of 3-dimensional complex left-symmetric algebras on the Lie algebras (b)-(e) according to the procedure

given in last section. As an explanation, we give a detailed and explicit demonstration for the left-symmetric algebras on \mathcal{H} , whereas we omit the length proof for other cases since the proof is similar.

3.1 The left-symmetric algebras on \mathcal{H}

The automorphism group of \mathcal{H} is

$$\text{Aut}(\mathcal{H}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix} \mid a_{11}a_{22} - a_{12}a_{21} \neq 0 \right\}. \quad (3.4)$$

It is easy to show that $f : \mathcal{H} \rightarrow gl(V)$ defined by equation (3.1) is a representation if and only if it satisfies the following conditions:

$$\begin{aligned} f_{11}(e_3) &= f_{22}(e_3) = f_{33}(e_3) = f_{21}(e_3) = f_{32}(e_3) = 0; \\ f_{21}(e_1)(f_{22} - f_{11})(e_2) + f_{21}(e_2)(f_{11} - f_{22})(e_1) &= 0; \\ f_{32}(e_1)(f_{33} - f_{22})(e_2) + f_{32}(e_2)(f_{22} - f_{33})(e_1) &= 0; \\ f_{31}(e_3)(f_{11} - f_{33})(e_1) &= f_{31}(e_3)(f_{11} - f_{33})(e_2) = 0; \\ f_{31}(e_3) &= f_{31}(e_1)(f_{33} - f_{11})(e_2) + f_{31}(e_2)(f_{11} - f_{33})(e_1) + f_{21}(e_1)f_{32}(e_2) - f_{21}(e_2)f_{32}(e_1). \end{aligned}$$

The 1-cocycle $C = (A_j(e_i))$ satisfies the following conditions:

$$\begin{aligned} A_3(e_3) &= 0, \quad -A_3(e_1)f_{33}(e_2) + A_3(e_2)f_{33}(e_1) = 0; \\ A_2(e_3)f_{22}(e_1) &= A_2(e_3)f_{22}(e_2) = 0; \\ A_2(e_1)f_{22}(e_2) - A_2(e_2)f_{22}(e_1) + A_3(e_1)f_{32}(e_2) - A_3(e_2)f_{32}(e_1) &= -A_2(e_3); \\ -A_1(e_3)f_{11}(e_1) - A_2(e_3)f_{21}(e_1) + A_3(e_1)f_{31}(e_3) &= 0; \\ -A_1(e_3)f_{11}(e_2) - A_2(e_3)f_{21}(e_2) + A_3(e_2)f_{31}(e_3) &= 0; \\ A_1(e_1)f_{11}(e_2) - A_1(e_2)f_{11}(e_1) + A_2(e_1)f_{21}(e_2) - A_2(e_2)f_{21}(e_1) + A_3(e_1)f_{31}(e_2) - A_3(e_2)f_{31}(e_1) \\ &= -A_1(e_3). \end{aligned}$$

Proposition 3.1 If $f_{31}(e_3) \neq 0$, then the representation must be equivalent to one of the following cases:

$$(AI) \quad f(e_1) = \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, \quad f(e_2) = \begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 0 & f_{11}(e_2) & 0 \\ 0 & 1 & f_{11}(e_2) \end{pmatrix}, \quad f(e_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof Since $f_{31}(e_3) \neq 0$, we can let $f_{31}(e_3) = 1$ through

$$v_1 \rightarrow f_{31}(e_3)v_1, v_2 \rightarrow v_2, v_3 \rightarrow v_3.$$

Thus, by the equations of f , we know that $f_{11} = f_{33}$ and $f_{32}(e_2)f_{21}(e_1) - f_{32}(e_1)f_{21}(e_2) = 1$. We claim that $f_{11} = f_{22}$. Otherwise, we can suppose $f_{11}(e_1) \neq f_{22}(e_1)$. Then through

$$v_1 \rightarrow v_1, v_2 \rightarrow v_2 - \frac{f_{21}(e_1)}{(f_{11} - f_{22})(e_1)}v_1, v_3 \rightarrow v_3 - \frac{f_{32}(e_1)}{(f_{22} - f_{11})(e_1)}v_2$$

we can let $f_{21}(e_1) = f_{32}(e_1) = 0$, which is a contradiction. On the other hand, we can let $f_{31}(e_1) = f_{31}(e_2) = 0$ through the following transformation which is in $\text{Aut}(\mathcal{H})$:

$$e_1 \rightarrow e_1 - f_{31}(e_1)e_3, e_2 \rightarrow e_2 - f_{31}(e_2)e_3, e_3 \rightarrow e_3.$$

Since f_{21} and f_{32} can not be zero, without losing generality, we suppose $f_{21}(e_1) \neq 0$. Thus by

$$e_1 \rightarrow e_1, e_2 \rightarrow e_2 - \frac{f_{21}(e_2)}{f_{21}(e_1)}e_1, e_3 \rightarrow e_3,$$

we can let $f_{21}(e_2) = 0$. Then $f_{21}(e_1)f_{32}(e_2) = 1$. By

$$v_1 \rightarrow v_1, v_2 \rightarrow f_{32}(e_2)v_2, v_3 \rightarrow v_3,$$

we can let $f_{32}(e_2) = f_{21}(e_1) = 1$. Finally, we can get (AI) by

$$e_1 \rightarrow e_1 - f_{32}(e_1)e_2, e_2 \rightarrow e_2, e_3 \rightarrow e_3. \quad \square$$

Proposition 3.2 For a representation given in the case (AI), there exist bijective 1-cocycles if and only if it is equivalent to one of the following cases:

(AI-1) $f_{11}(e_1) = 1, f_{11}(e_2) = 0$. There is only one bijective 1-cocycle up to equivalence

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow (H - 1) \begin{pmatrix} e_1 & e_2 + e_3 & e_3 \\ e_2 & 0 & 0 \\ e_3 & 0 & 0 \end{pmatrix}.$$

(AI-2) $f_{11}(e_1) = f_{11}(e_2) = 1$. There is only one bijective 1-cocycle up to equivalence

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow (H - 2)' \begin{pmatrix} e_1 & e_2 + e_3 & e_3 \\ e_2 & 2e_2 - e_1 & e_3 \\ e_3 & e_3 & 0 \end{pmatrix} \cong (H - 2) \begin{pmatrix} e_1 & e_2 + e_3 & e_3 \\ e_2 & e_3 & 0 \\ e_3 & 0 & 0 \end{pmatrix}.$$

Proof For a representation given in (AI), the equations for $C = (A_j(e_i))$ reduce to the following equations:

$$A_3(e_3) = A_2(e_3) = 0; f_{11}(e_1)A_3(e_2) = f_{11}(e_2)A_3(e_1);$$

$$A_3(e_1) = A_2(e_2)f_{11}(e_1) - A_2(e_1)f_{11}(e_2) = A_1(e_3)f_{11}(e_1);$$

$$A_3(e_2) = A_1(e_3)f_{11}(e_2); A_1(e_3) = A_2(e_2) + A_1(e_2)f_{11}(e_1) - A_1(e_1)f_{11}(e_2).$$

If $f_{11}(e_1) = 0$, then $A_3(e_1) = 0$. If $f_{11}(e_2) \neq 0$, then $A_2(e_1) = 0$, which leads to $\det C = 0$. If $f_{11}(e_2) = 0$, then $A_3(e_2) = 0$, which also leads to $\det C = 0$.

If $f_{11}(e_1) \neq 0$, then we can let $f_{11}(e_1) = 1$ by

$$e_1 \rightarrow \frac{1}{f_{11}(e_1)}e_1, e_2 \rightarrow e_2, e_3 \rightarrow \frac{1}{f_{11}(e_1)}e_3; v_1 \rightarrow \frac{1}{f_{11}(e_1)}v_1, v_2 \rightarrow v_2, v_3 \rightarrow v_3.$$

If $f_{11}(e_2) = 0$, this is just the case (AI-1). At the time, the corresponding bijective 1-cocycles are given by $C = \begin{pmatrix} A_1(e_1) & A_2(e_1) & A_3(e_1) \\ 0 & A_3(e_1) & 0 \\ A_3(e_1) & 0 & 0 \end{pmatrix}$ with $A_3(e_1) \neq 0$. The corresponding left-symmetric algebras are $\begin{pmatrix} e_1 + \frac{A_2(e_1)}{A_3(e_1)}e_3 & e_2 + e_3 & e_3 \\ e_2 & 0 & 0 \\ e_3 & 0 & 0 \end{pmatrix}$. However, they are isomorphic to (H-1) through

$$e_1 \rightarrow e_1 - \frac{A_2(e_1)}{A_3(e_1)}e_3, e_2 \rightarrow e_2, e_3 \rightarrow e_3,$$

which can be given by $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ directly.

If $f_{11}(e_2) \neq 0$, then we can let $f_{11}(e_2) = 1$ by

$$e_1 \rightarrow e_1, e_2 \rightarrow \frac{1}{f_{11}(e_2)}e_2, e_3 \rightarrow \frac{1}{f_{11}(e_2)}e_3; v_1 \rightarrow v_1, v_2 \rightarrow v_2, v_3 \rightarrow f_{11}(e_2)v_3,$$

which is the case (AI-2). The corresponding bijective 1-cocycles are given by

$$C = \begin{pmatrix} A_1(e_1) & A_2(e_1) & A_3(e_1) \\ A_1(e_1) - A_2(e_1) & A_2(e_1) + A_3(e_1) & A_3(e_1) \\ A_3(e_1) & 0 & 0 \end{pmatrix}$$

with $A_3(e_1) \neq 0$. The corresponding left-symmetric algebras are

$$\begin{pmatrix} e_1 + \frac{A_2(e_1)}{A_3(e_1)}e_3 & e_2 + \frac{A_2(e_1) + A_3(e_1)}{A_3(e_1)}e_3 & e_3 \\ e_2 + \frac{A_2(e_1)}{A_3(e_1)}e_3 & 2e_2 - e_1 + \frac{A_2(e_1)}{A_3(e_1)}e_3 & e_3 \\ e_3 & e_3 & 0 \end{pmatrix}.$$

However, they are isomorphic to (H-2)' through

$$e_1 \rightarrow e_1 - \frac{A_2(e_1)}{A_3(e_1)}e_3, e_2 \rightarrow e_2, e_3 \rightarrow e_3,$$

which can be given by $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ directly. Notice that (H-2)' \cong (H-2) through

$$e_1 \rightarrow e_1, e_2 \rightarrow e_1 - e_2, e_3 \rightarrow -e_3.$$

□

Proposition 3.3 If $f_{31}(e_3) = 0$, then the equivalent classes of the representations of \mathcal{H} are divided into the following cases (maybe there are equivalent classes in the same case, but not equivalent in different cases):

$$\begin{aligned}
(\text{BI}) \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{22}(e_1) & 0 \\ 0 & 0 & f_{33}(e_1) \end{pmatrix}, f(e_2) = \begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 0 & f_{22}(e_2) & 0 \\ 0 & 0 & f_{33}(e_2) \end{pmatrix}, f(e_3) = 0. \\
(\text{BII}) \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{33}(e_1) \end{pmatrix}, f(e_2) = \begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 1 & f_{11}(e_2) & 0 \\ 0 & 0 & f_{33}(e_2) \end{pmatrix}, f(e_3) = 0. \\
(\text{BIII}) \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = \begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 1 & f_{11}(e_2) & 0 \\ 0 & 1 & f_{11}(e_2) \end{pmatrix}, f(e_3) = 0. \\
(\text{BIV}) \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = \begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 0 & f_{11}(e_2) & 0 \\ 1 & 0 & f_{11}(e_2) \end{pmatrix}, f(e_3) = 0. \\
(\text{BV}) \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 1 & f_{11}(e_1) \end{pmatrix}, f(e_2) = \begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 0 & f_{11}(e_2) & 0 \\ 1 & 0 & f_{11}(e_2) \end{pmatrix}, f(e_3) = 0. \\
(\text{BVI}) \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 1 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = \begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 1 & f_{11}(e_2) & 0 \\ 0 & 1 & f_{11}(e_2) \end{pmatrix}, f(e_3) = 0.
\end{aligned}$$

Proof We give the sketch of proof here. The detailed discussion is as the same as the discussion in the case (AI). Since $f(e_3) = 0$, we only need to consider $f(e_1)$ and $f(e_2)$. Moreover, there is a kind of symmetry between $f(e_1)$ and $f(e_2)$ since $e_1 \rightarrow e_2, e_2 \rightarrow e_1, e_3 \rightarrow -e_3$ is in $\text{Aut}(\mathcal{H})$. Hence we only need to consider the Jordan canonical forms of $f(e_1)$.

$f(e_1)$ is diagonalized. Then the equations of f reduce to

$$f_{21}(e_2)(f_{11} - f_{22})(e_1) = f_{32}(e_2)(f_{22} - f_{33})(e_1) = f_{31}(e_2)(f_{11} - f_{33}) = 0.$$

Thus we can consider the Jordan canonical form of $f(e_2)$. First of all, $f(e_2)$ is also diagonalized. This is the case (BI). Secondly, $f(e_2)$ is the type $\begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 1 & f_{11}(e_2) & 0 \\ 0 & 0 & f_{33}(e_2) \end{pmatrix}$, then $f_{11}(e_1) = f_{22}(e_1)$, which is the case (BII). For the other positions of Jordan blocks of $f(e_2)$, it is easy to show that they are isomorphic. Finally, $f(e_2)$ is the type $\begin{pmatrix} f_{11}(e_2) & 0 & 0 \\ 1 & f_{11}(e_2) & 0 \\ 0 & 1 & f_{33}(e_2) \end{pmatrix}$, then $f_{11}(e_1) = f_{22}(e_1) = f_{33}(e_1)$. This is the case (BIII).

$f(e_1)$ is the type $\begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{33}(e_1) \end{pmatrix}$. Then from the equations of f , we can get

$$f_{32}(e_2) = 0, \quad f_{31}(e_2)(f_{11} - f_{33})(e_1) = 0.$$

We can let $f_{21}(e_2) = 0$ through

$$e_1 \rightarrow e_1, e_2 \rightarrow e_2 - f_{21}(e_2)e_1, e_3 \rightarrow e_3.$$

If $f_{31}(e_2) = 0$, by symmetry of e_1, e_2 , it is equivalent to the case (BII). If $f_{31}(e_2) \neq 0$, then $f_{11}(e_1) = f_{33}(e_1)$ and moreover, we can let $f_{31}(e_2) = 1$ by

$$e_1 \rightarrow e_1, e_2 \rightarrow \frac{1}{f_{31}(e_2)}e_2, e_3 \rightarrow \frac{1}{f_{31}(e_2)}e_3,$$

which is just the case (BIV). Similarly, for other positions of the Jordan blocks of $f(e_1)$ with the same type such as $f(e_1) = \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{22}(e_1) & 0 \\ 0 & 1 & f_{22}(e_1) \end{pmatrix}$ and $f(e_1) = \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{22}(e_1) & 0 \\ 1 & 0 & f_{11}(e_1) \end{pmatrix}$, we can get the case (BV) and (BVI) respectively.

$f(e_1)$ is the type $\begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 1 & f_{11}(e_1) \end{pmatrix}$. Then from the equations of f , we know $f_{11}(e_2) = f_{22}(e_2) = f_{33}(e_2), f_{32}(e_2) = f_{21}(e_2)$.

We can let $f_{32}(e_2) = f_{21}(e_2) = 0$ by

$$e_1 \rightarrow e_1, e_2 \rightarrow e_2 - f_{32}(e_2)e_1, e_3 \rightarrow e_3.$$

Then it is equivalent to the case (BIII) if $f_{31}(e_2) = 0$ or the case (BVI) if $f_{31}(e_2) \neq 0$. \square

Proposition 3.4 For a representation of \mathcal{H} given in the above cases respectively, there exist bijective 1-cocycles if and only if it is equivalent to one of the following corresponding cases: (we give the classification of 1-cocycles up to equivalence and the corresponding left-symmetric algebras respectively)

Case (BI): there does not exist any bijective 1-cocycle;

Case (BII): (BII-1) $f_{11}(e_1) = 0, f_{33}(e_1) = 1, f_{11}(e_2) = f_{33}(e_2) = 0$.

$$C = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow (H-3)' \begin{pmatrix} e_1 + e_2 & 0 & 0 \\ -e_3 & e_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong (H-3) \begin{pmatrix} e_1 & e_3 & 0 \\ 0 & e_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(BII-2) $f_{11}(e_1) = f_{33}(e_1) = 0, f_{11}(e_2) = 0, f_{33}(e_2) = 1$.

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \Rightarrow (H-4)' \begin{pmatrix} 0 & 0 & 0 \\ -e_3 & e_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong (H-4) \begin{pmatrix} e_1 & e_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(BII-3) $f_{11}(e_1) = f_{33}(e_1) = f_{11}(e_2) = f_{33}(e_2) = 0$.

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \Rightarrow (H-5) \begin{pmatrix} 0 & 0 & 0 \\ -e_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case (BIII): $f_{11}(e_1) = f_{11}(e_2) = 0$:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \implies (\text{H} - 6) \begin{pmatrix} 0 & 0 & 0 \\ -e_3 & e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case (BIV): $f_{11}(e_1) = f_{11}(e_2) = 0$:

$$C_\lambda = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & \lambda \\ 1 & 0 & 0 \end{pmatrix}, \lambda \neq 0, \implies (\text{H} - 7)_\lambda \begin{pmatrix} e_3 & e_3 & 0 \\ 0 & \lambda e_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda \neq 0.$$

$$C = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix} \implies (\text{H} - 8) \begin{pmatrix} 0 & \frac{1}{2}e_3 & 0 \\ -\frac{1}{2}e_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case (BV): $f_{11}(e_1) = f_{11}(e_2) = 0$:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \implies (\text{H} - 9) \begin{pmatrix} 0 & e_3 & 0 \\ 0 & e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Case (BVI): $f_{11}(e_1) = f_{11}(e_2) = 0$:

$$C_\lambda = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \lambda \\ \lambda - 1 & 0 & 0 \end{pmatrix}, \lambda \neq 0, 1 \implies (\text{H} - 10)_\lambda \begin{pmatrix} 0 & \frac{\lambda}{\lambda-1}e_3 & 0 \\ \frac{1}{\lambda-1}e_3 & \lambda e_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda \neq 0, 1.$$

Proof It is as the same as the proof of Proposition 3.2 with the computation case by case. \square

Moreover, through a direct computation, we know

Proposition 3.5 With the notations as above, among the left-symmetric algebras on \mathcal{H} , we have

- a) Associative algebras: (H-5), (H-7) $_\lambda$ ($\lambda \neq 0$), (H-8);
- b) Transitive left-symmetric algebras: (H-5), (H-6), (H-7) $_\lambda$ ($\lambda \neq 0$), (H-8), (H-9), (H-10) $_\lambda$ ($\lambda \neq 0, 1$);
- c) Novikov algebras: (H-1), (H-2), (H-5), (H-6), (H-7) $_\lambda$ ($\lambda \neq 0$), (H-8), (H-9), (H-10) $_\lambda$ ($\lambda \neq 0, 1$);
- d) Bi-symmetric algebras: (H-5), (H-6), (H-7) $_\lambda$ ($\lambda \neq 0$), (H-8), (H-9), (H-10) $_\lambda$ ($\lambda \neq 0, 1$);
- e) There is not any simple left-symmetric algebra on \mathcal{H} .

3.2 The left-symmetric algebras on \mathcal{N}

The automorphism group of \mathcal{N} is

$$\text{Aut}(\mathcal{N}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \mid a_{11}a_{22} \neq 0 \right\}. \quad (3.5)$$

Proposition 3.6 The equivalent classes of the representations of \mathcal{N} are divided into the following cases (maybe there are equivalent classes in the same case, but not equivalent in different cases):

Set

$$B_J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_J = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} \text{(AI)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{22}(e_1) & 0 \\ 0 & 0 & f_{33}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(AII)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{33}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(AIII)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 1 & f_{11}(e_3) \end{pmatrix}. \\ \text{(AIV)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{33}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{11}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(AV)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{11}(e_3) & 0 \\ 1 & 0 & f_{11}(e_3) \end{pmatrix}. \\ \text{(AVI)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 1 & f_{11}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{11}(e_3) & 0 \\ 1 & 0 & f_{11}(e_3) \end{pmatrix}. \\ \text{(AVII)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 1 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 1 & f_{11}(e_3) \end{pmatrix}. \\ \text{(AVIII)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 1 & f_{11}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{11}(e_3) & 0 \\ 0 & 0 & f_{11}(e_3) \end{pmatrix}. \\ \text{(AIX)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 1 & f_{11}(e_1) \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{11}(e_3) & 0 \\ 1 & 0 & f_{11}(e_3) \end{pmatrix}. \\ \text{(BI)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{22}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = B_J, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(BII)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = B_J, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 1 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(BIII)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = B_J, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 0 & f_{33}(e_3) & 0 \\ 0 & 1 & f_{33}(e_3) \end{pmatrix}. \\ \text{(BIV)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 1 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = B_J, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \end{aligned}$$

$$\begin{aligned}
\text{(BV)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 1 & f_{11}(e_1) \end{pmatrix}, f(e_2) = B_J, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 0 & f_{33}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
\text{(CI)} \quad f(e_1) &= \begin{pmatrix} f_{11}(e_1) & 0 & 0 \\ 0 & f_{11}(e_1) & 0 \\ 0 & 0 & f_{11}(e_1) \end{pmatrix}, f(e_2) = C_J, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 2 & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}.
\end{aligned}$$

Proposition 3.7 For a representation of \mathcal{N} given in the above cases respectively, there exist bijective 1-cocycles if and only if it is equivalent to one of the following corresponding cases:

Case (AI): (AI-1) $f_{11}(e_1) = f_{22}(e_1) = f_{33}(e_1) = 0, f_{11}(e_3) = 1, f_{22}(e_3) = 0, f_{33}(e_3) = \lambda, \lambda \in \mathbf{C}$.

$$\forall \lambda \in \mathbf{C}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (N-1)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & \lambda e_3 \end{pmatrix}, \lambda \in \mathbf{C}.$$

(AI-2) $f_{11}(e_1) = 0, f_{22}(e_1) = 1, f_{33}(e_1) = \lambda, f_{11}(e_3) = 1, f_{22}(e_3) = 0, f_{33}(e_3) = \mu, \lambda \neq 0, \mu \neq 0$.

$$\forall \lambda \neq 0, \mu \neq 0, \quad C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{\mu}{\lambda} \end{pmatrix} \implies (N-2)_{\lambda, \mu} = \begin{pmatrix} e_1 + \frac{\lambda(\lambda-1)}{\mu} e_3 & 0 & \lambda e_3 \\ 0 & 0 & 0 \\ \lambda e_3 & e_2 & \mu e_3 \end{pmatrix}, \lambda \neq 0, \mu \neq 0.$$

(AI-3) $f_{11}(e_1) = 0, f_{22}(e_1) = 1, f_{33}(e_1) = 0, f_{11}(e_3) = 1, f_{22}(e_3) = 0, f_{33}(e_3) = \mu, \mu \neq 0$.

$$\forall \mu \neq 0, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (N-3)_\mu = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & \mu e_3 \end{pmatrix}, \mu \neq 0.$$

(AI-4) $f_{11}(e_1) = 0, f_{22}(e_1) = 1, f_{33}(e_1) = 0, f_{11}(e_3) = 1, f_{22}(e_3) = f_{33}(e_3) = 0$.

$$C_\lambda = \begin{pmatrix} 0 & 1 & -\lambda \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda \in \mathbf{C} \implies (N-4)_\lambda = \begin{pmatrix} e_1 + \lambda e_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}, \lambda \in \mathbf{C}.$$

Case (AII): (AII-1) $f_{11}(e_1) = f_{33}(e_1) = 0, f_{11}(e_3) = 0, f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \implies (N-5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_1 \end{pmatrix}.$$

(AII-2) $f_{11}(e_1) = f_{33}(e_1) = 0, f_{11}(e_3) = 1, f_{33}(e_3) = 0$.

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies (N-6) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_3 + e_2 \end{pmatrix}.$$

(AII-3) $f_{11}(e_1) = 0, f_{33}(e_1) = 1, f_{11}(e_3) = 1, f_{33}(e_3) = 0$.

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies (N-7) = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & e_3 + e_2 \end{pmatrix}.$$

(AII-4) $f_{11}(e_1) = 1, f_{33}(e_1) = 0, f_{11}(e_3) = 0, f_{33}(e_3) = 1.$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \implies (N-8) = \begin{pmatrix} e_1 & 0 & e_3 \\ 0 & 0 & 0 \\ e_3 & e_2 & 0 \end{pmatrix}.$$

Case (AIII): there does not exist any bijective 1-cocycle;

Case (AIV): (AIV-1) $f_{11}(e_1) = f_{33}(e_1) = 0, f_{11}(e_3) = \lambda, f_{33}(e_3) = 1, \lambda \neq 0.$

$$\forall \lambda \neq 0, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \lambda & 0 \end{pmatrix} \implies (N-9)_\lambda = \begin{pmatrix} 0 & 0 & \lambda e_1 \\ 0 & 0 & 0 \\ \lambda e_1 & e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0$$

(AIV-2) $f_{11}(e_1) = f_{33}(e_1) = 0, f_{11}(e_3) = 0, f_{33}(e_3) = 1.$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \implies (N-10) = \begin{pmatrix} e_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}.$$

(AIV-3) $f_{11}(e_1) = f_{33}(e_1) = 0, f_{11}(e_3) = 1, f_{33}(e_3) = 0.$

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies (N-11) = \begin{pmatrix} 0 & 0 & e_2 \\ 0 & 0 & 0 \\ e_2 & e_2 & e_3 \end{pmatrix}.$$

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \implies (N-12)' = \begin{pmatrix} 0 & 0 & e_2 \\ 0 & 0 & 0 \\ e_2 & e_2 & e_3 + e_2 - e_1 \end{pmatrix} \cong (N-12) = \begin{pmatrix} 0 & 0 & e_2 \\ 0 & 0 & 0 \\ e_2 & e_2 & e_3 - e_2 \end{pmatrix}.$$

(AIV-4) $f_{11}(e_1) = 0, f_{33}(e_1) = 1, f_{11}(e_3) = 1, f_{33}(e_3) = \lambda, \lambda \in \mathbf{C}.$

$$\begin{aligned} \forall \lambda \in \mathbf{C}, C = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & \lambda \end{pmatrix} \implies (N-13)'_\lambda &= \begin{pmatrix} e_1 - e_2 & 0 & \lambda e_1 + (1-\lambda)e_2 \\ 0 & 0 & 0 \\ \lambda e_1 + (1-\lambda)e_2 & e_2 & e_3 + (\lambda^2 - \lambda)(e_1 - e_2) \end{pmatrix} \\ &\cong (N-13)_\lambda = \begin{pmatrix} e_1 - e_2 & 0 & e_2 \\ 0 & 0 & 0 \\ e_2 & e_2 & e_3 - \lambda e_2 \end{pmatrix}, \lambda \in \mathbf{C} \end{aligned}$$

(AIV-5) $f_{11}(e_1) = 1, f_{33}(e_1) = 0, f_{11}(e_3) = \lambda, f_{33}(e_3) = 1, \lambda \neq 0.$

$$\begin{aligned} \forall \lambda \neq 0, C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda & \lambda & 0 \end{pmatrix} \implies (N-14)'_\lambda &= \begin{pmatrix} 2e_1 - \frac{1}{\lambda}e_3 & 0 & \lambda e_1 \\ 0 & 0 & 0 \\ \lambda e_1 & e_2 & \lambda e_3 \end{pmatrix} \\ &\cong (N-14)_\lambda = \begin{pmatrix} e_1 - \frac{1}{\lambda}e_3 & 0 & e_3 \\ 0 & 0 & 0 \\ e_3 & e_2 & 0 \end{pmatrix}, \lambda \neq 0 \end{aligned}$$

Case (AV): there does not exist any bijective 1-cocycle;

Case (AVI): $f_{11}(e_1) = 0, f_{11}(e_3) = 1.$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (N-15) = \begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & e_2 & e_3 + e_2 \end{pmatrix}.$$

Case (AVII): there does not exist any bijective 1-cocycle;

Case (AVIII): $f_{11}(e_1) = 0, f_{11}(e_3) = 1$.

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (N - 16) = \begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & e_2 & e_3 \end{pmatrix}.$$

Case (AIX): $f_{11}(e_1) = 0, f_{11}(e_3) = 1$.

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (N - 17) = \begin{pmatrix} e_2 & 0 & e_1 \\ 0 & 0 & 0 \\ e_1 & e_2 & e_3 + e_2 \end{pmatrix}.$$

Case (BI): (BI-1) $f_{11}(e_1) = f_{22}(e_1) = 0, f_{22}(e_3) = 0, f_{33}(e_3) = \lambda, \lambda \neq 0$.

$$\forall \lambda \neq 0, C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \implies (N - 18)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda e_2 \\ 0 & (\lambda + 1)e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \implies (N - 19) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e_2 \\ 0 & 0 & -e_3 + e_2 \end{pmatrix};$$

when $\lambda = 1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \implies (N - 20) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & e_2 & 2e_3 \end{pmatrix}.$$

(BI-2) $f_{11}(e_1) = f_{22}(e_1) = 0, f_{22}(e_3) = 1, f_{33}(e_3) = -1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (N - 21) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ 0 & e_2 + e_1 & -e_3 \end{pmatrix};$$

(BI-3) $f_{11}(e_1) = 1, f_{22}(e_1) = \lambda, f_{22}(e_3) = \mu, f_{33}(e_3) = 0, \lambda \neq 0, \mu \neq 0$.

$$\forall \lambda \neq 0, \mu \neq 0, C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & \frac{\mu}{\lambda} & 0 \end{pmatrix} \implies (N - 22)_{\lambda, \mu} = \begin{pmatrix} e_1 + \frac{\lambda(\lambda-1)}{\mu}e_3 & e_2 & \lambda e_3 \\ e_2 & 0 & 0 \\ \lambda e_3 & e_2 & \mu e_3 \end{pmatrix}, \lambda \neq 0, \mu \neq 0.$$

(BI-4) $f_{11}(e_1) = 1, f_{22}(e_1) = 0, f_{22}(e_3) = \mu, f_{33}(e_3) = 0, \mu \neq 0, 1$.

$$\forall \mu \neq 0, 1, C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies (N - 23)_\mu = \begin{pmatrix} e_1 & e_2 & 0 \\ e_2 & 0 & 0 \\ 0 & e_2 & \mu e_3 \end{pmatrix}, \mu \neq 0, 1.$$

(BI-5) $f_{11}(e_1) = 1, f_{22}(e_1) = 0, f_{22}(e_3) = f_{33}(e_3) = 0$.

$$C_\lambda = \begin{pmatrix} 0 & -\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \lambda \in \mathbf{C} \implies (N - 24)_\lambda = \begin{pmatrix} e_1 + \lambda e_3 & e_2 & 0 \\ e_2 & 0 & 0 \\ 0 & e_2 & 0 \end{pmatrix}, \lambda \in \mathbf{C}.$$

$$(BI-6) \ f_{11}(e_1) = 1, f_{22}(e_1) = 0, f_{22}(e_3) = 1, f_{33}(e_3) = 0.$$

$$C_\lambda = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}, \lambda \in \mathbf{C} \implies (N-25)_\lambda = \begin{pmatrix} e_1 & e_2 + \lambda e_3 & 0 \\ e_2 + \lambda e_3 & 0 & 0 \\ 0 & e_2 & e_3 \end{pmatrix}, \lambda \in \mathbf{C}.$$

$$(BI-7) \ f_{11}(e_1) = 0, f_{22}(e_1) = 1, f_{22}(e_3) = 0, f_{33}(e_3) = \lambda, \lambda \neq 0.$$

$$\forall \lambda \neq 0 \ C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \implies (N-26)_\lambda = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & \lambda e_2 \\ 0 & (\lambda+1)e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is three additional equivalent classes:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \implies (N-27) = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & 0 & -e_2 \\ 0 & 0 & -e_3 + e_2 \end{pmatrix};$$

$$C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \implies (N-28) = \begin{pmatrix} e_1 + e_2 & 0 & 0 \\ 0 & 0 & -e_2 \\ 0 & 0 & -e_3 \end{pmatrix};$$

$$C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \implies (N-29) = \begin{pmatrix} e_1 + e_2 & 0 & 0 \\ 0 & 0 & -e_2 \\ 0 & 0 & -e_3 + e_2 \end{pmatrix};$$

when $\lambda = 1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \implies (N-30) = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_3 & 0 \\ 0 & e_2 & 2e_3 \end{pmatrix}.$$

Case (BII): $f_{11}(e_1) = 1, f_{33}(e_3) = -1$.

$$C = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \implies (N-31) = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_2 & 0 & 0 \\ e_3 & e_2 & e_3 + e_2 \end{pmatrix}.$$

Case (BIII): $f_{11}(e_1) = 1, f_{11}(e_3) = 0$.

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies (N-32) = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_2 & 0 & 0 \\ e_3 & e_2 & 0 \end{pmatrix}.$$

Case (BIV): (BIV-1) $f_{11}(e_1) = 0, f_{33}(e_3) = -1$.

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \implies (N-33) = \begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & -e_2 \\ 0 & 0 & -e_3 \end{pmatrix};$$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \implies (N-34) = \begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & -e_2 \\ 0 & 0 & -e_3 + e_2 \end{pmatrix}.$$

$$(BIV-2) \ f_{11}(e_1) = 0, f_{33}(e_3) = 0.$$

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \implies (N-35) = \begin{pmatrix} 0 & e_3 & 0 \\ e_3 & 0 & 0 \\ 0 & e_2 & e_3 \end{pmatrix}.$$

$$(BIV-3) \ f_{11}(e_1) = 0, f_{33}(e_3) = 1.$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \implies (N-36) = \begin{pmatrix} 0 & 0 & 2e_1 \\ 0 & e_1 & 0 \\ 2e_1 & e_2 & 2e_3 \end{pmatrix}.$$

$$(BIV-4) \ f_{11}(e_1) = 1, f_{11}(e_3) = \lambda, \lambda \in \mathbf{C}.$$

$$\begin{aligned} \forall \lambda \in \mathbf{C}, \ C &= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -(\lambda+1) & \lambda+1 & \lambda \end{pmatrix} \implies \\ (N-37)'_{\lambda} &= \begin{pmatrix} e_1+e_2 & e_2 & e_3+(1+\lambda)e_2 \\ e_2 & 0 & \lambda e_2 \\ e_3+(1+\lambda)e_2 & (\lambda+1)e_2 & (2\lambda+1)e_3 - \lambda(\lambda+1)(e_1-e_2) \end{pmatrix} \\ &\cong (N-37)_{\lambda} = \begin{pmatrix} e_1+e_2 & e_2 & e_2+e_3 \\ e_2 & 0 & 0 \\ e_2+e_3 & e_2 & e_3-\lambda e_2 \end{pmatrix}, \lambda \in \mathbf{C} \end{aligned}$$

$$\text{Case (BV): } (BV-1) \ f_{11}(e_1) = 0, f_{33}(e_3) = \lambda, \lambda \neq 0.$$

$$\forall \lambda \neq 0, \ C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \implies (N-38)_{\lambda} = \begin{pmatrix} 0 & 0 & \lambda e_1 \\ 0 & 0 & \lambda e_2 \\ \lambda e_1 & (\lambda+1)e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is three additional equivalent classes:

$$\begin{aligned} C &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \implies (N-39) = \begin{pmatrix} 0 & 0 & -e_1 \\ 0 & 0 & -e_2 \\ -e_1 & 0 & -e_3+e_2 \end{pmatrix}; \\ C &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \implies (N-40) = \begin{pmatrix} 0 & 0 & -e_1+e_2 \\ 0 & 0 & -e_2 \\ -e_1+e_2 & 0 & -e_3 \end{pmatrix}; \\ C &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \implies (N-41) = \begin{pmatrix} 0 & 0 & -e_1+e_2 \\ 0 & 0 & -e_2 \\ -e_1+e_2 & 0 & -e_3+e_2 \end{pmatrix}; \end{aligned}$$

when $\lambda = 1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (N-42) = \begin{pmatrix} 0 & 0 & e_1 \\ 0 & 0 & e_2-e_1 \\ e_1 & 2e_2-e_1 & e_3 \end{pmatrix}.$$

$$(BV-2) \ f_{11}(e_1) = 1, f_{33}(e_3) = \lambda, \lambda \neq 0.$$

$$\forall \lambda \neq 0, \ C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -\lambda & \lambda \end{pmatrix} \implies (N-43)_{\lambda} = \begin{pmatrix} e_1 - \frac{1}{\lambda}e_3 & e_2 & e_3 \\ e_2 & 0 & 0 \\ e_3 & e_2 & 0 \end{pmatrix}, \lambda \neq 0.$$

Case (CI): (CI-1) $f_{11}(e_1) = 0, f_{33}(e_3) = -2$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \implies (N - 44) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & -2e_2 \\ 0 & -e_2 & -2e_3 \end{pmatrix}.$$

(CI-2) $f_{11}(e_1) = 1, f_{33}(e_3) = 0$.

$$C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \implies (N - 45) = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_2 & e_3 & 0 \\ e_3 & e_2 & 2e_3 \end{pmatrix}.$$

Remark It is easy to see that we can extend the extent of some parameters appearing in above left-symmetric algebras:

- (1) $(N - 2)_{\lambda=0, \mu \neq 0} \cong (N - 3)_{\mu \neq 0}; (N - 3)_{\mu=0} \cong (N - 4)_{\lambda=0} \cong (N - 26)_{\lambda=0};$
- (2) $(N - 22)_{\lambda=0, \mu \neq 0, 1} \cong (N - 23)_{\mu \neq 0, 1}; (N - 22)_{\lambda=0, \mu=1} \cong (N - 23)_{\mu=1} \cong (N - 25)_{\lambda=0};$
 $(N - 23)_{\mu=0} \cong (N - 24)_{\lambda=0};$
- (3) $(N - 1)_{\mu=0} \cong (N - 9)_{\lambda=0} \cong (N - 18)_{\lambda=0} \cong (N - 38)_{\lambda=0}.$

Proposition 3.8 With the notations as above, among the left-symmetric algebras on \mathcal{N} , we have

- a) Associative algebras: $(N-1)_1, (N-3)_1, (N-9)_1, (N-18)_{-1}, (N-22)_{1,1}, (N-26)_{-1};$
- b) Transitive left-symmetric algebras: $(N-1)_0, (N-5), (N-10);$
- c) Novikov algebras: $(N-1)_0, (N-4)_0, (N-5), (N-18)_\lambda (\lambda \neq 0), (N-19), (N-26)_\lambda (\lambda \neq 0), (N-27),$
 $(N-38)_\lambda (\lambda \neq 0), (N-39);$
- d) Bi-symmetric algebras: $(N-1)_1, (N-3)_1, (N-6), (N-7), (N-9)_1, (N-15), (N-18)_{-1}, (N-19), (N-22)_{1,1}, (N-26)_{-1}; (N-27), (N-31), (N-38)_{-1}, (N-39).$
- e) There is not any simple left-symmetric algebra on \mathcal{N} . But $(N-30)$ is semisimple.

3.3 The left-symmetric algebras on \mathcal{D}_1

The automorphism group of \mathcal{D}_1 is

$$\text{Aut}(\mathcal{D}_1) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \mid a_{11}a_{22} - a_{12}a_{21} \neq 0 \right\}. \quad (3.6)$$

Proposition 3.9 The equivalent classes of the representations of \mathcal{D}_1 are divided into the following cases:

$$(AI) f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}.$$

$$\begin{aligned}
(\text{AII}) \quad & f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{AIII}) \quad & f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 1 & f_{11}(e_3) \end{pmatrix}. \\
(\text{AIV}) \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{22}(e_3) + 1 & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{AV}) \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) & 0 & 0 \\ 0 & f_{33}(e_3) - 1 & 0 \\ 1 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{AVI}) \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 1 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{AVII}) \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 2 & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{BI}) \quad & f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 0 & f_{33}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{BII}) \quad & f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}.
\end{aligned}$$

Proposition 3.10 For a representation of \mathcal{D}_1 given in the above cases respectively, there exist

bijjective 1-cocycles if and only if it is equivalent to one of the following corresponding cases:

Case (AI): $f_{11}(e_3) = f_{22}(e_3) = 1, f_{33}(e_3) = \lambda, \lambda \in \mathbf{C}$.

$$\forall \lambda \in \mathbf{C}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (\bar{D}_1 - 1)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_2 & \lambda e_3 \end{pmatrix}, \lambda \in \mathbf{C}.$$

Case (AII): $f_{11}(e_3) = f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies (\bar{D}_1 - 2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_2 & e_3 + e_2 \end{pmatrix}.$$

Case (AIII): there does not exist any bijjective 1-cocycle;

Case (AIV): $f_{22}(e_3) = \lambda, f_{33}(e_3) = 1, \lambda \neq 0$.

$$\forall \lambda \neq 0, \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix} \implies (\bar{D}_1 - 3)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda e_2 \\ e_1 & (\lambda + 1)e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \implies (\bar{D}_1 - 4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e_2 \\ e_1 & 0 & -e_3 + e_2 \end{pmatrix};$$

when $\lambda = 1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow (\bar{D}_1 - 5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_3 & 0 \\ e_1 & e_2 & 2e_3 \end{pmatrix}.$$

Case (AV): $f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow (\bar{D}_1 - 6) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ e_1 & e_1 + e_2 & 0 \end{pmatrix}.$$

Case (AVI): $f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow (\bar{D}_1 - 7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_2 \\ e_1 & 2e_2 & e_3 + e_1 \end{pmatrix}.$$

Case (AVII): $f_{33}(e_3) = -1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow (\bar{D}_1 - 8) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & -e_2 \\ e_1 & 0 & -e_3 \end{pmatrix}.$$

Case (BI): $f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow (\bar{D}_1 - 9) = \begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_1 & 0 \\ 2e_1 & e_2 & e_3 \end{pmatrix};$$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow (\bar{D}_1 - 10) = \begin{pmatrix} e_3 & 0 & 0 \\ 0 & e_3 & 0 \\ e_1 & e_2 & 2e_3 \end{pmatrix}.$$

Case (BII): $f_{33}(e_3) = \lambda, \lambda \neq 0$.

$$\forall \lambda \neq 0, C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow (\bar{D}_1 - 11)_\lambda = \begin{pmatrix} 0 & 0 & \lambda e_1 \\ 0 & 0 & \lambda e_2 \\ (\lambda + 1)e_1 & (\lambda + 1)e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \Rightarrow (\bar{D}_1 - 12) = \begin{pmatrix} 0 & 0 & -e_1 \\ 0 & 0 & -e_2 \\ 0 & 0 & -e_3 + e_2 \end{pmatrix}.$$

Remark It is easy to see that we can extend the extent of the some parameters appearing in above left-symmetric algebras:

$$(\bar{D}_1 - 1)_{\lambda=0} \cong (\bar{D}_1 - 3)_{\lambda=0} \cong (\bar{D}_1 - 11)_{\lambda=0}.$$

Proposition 3.11 With the notations as above, among the left-symmetric algebras on \mathcal{D}_1 , we have

- a) Associative algebras: $(\bar{\mathcal{D}}_1-1)_1, (\bar{\mathcal{D}}_1-11)_{-1}$;
- b) Transitive left-symmetric algebras: $(\bar{\mathcal{D}}_1-1)_0, (\bar{\mathcal{D}}_1-6)$;
- c) Novikov algebras: $(\bar{\mathcal{D}}_1-1)_0, (\bar{\mathcal{D}}_1-11)_\lambda (\lambda \neq 0), (\bar{\mathcal{D}}_1-12)$;
- d) Bi-symmetric algebras: $(\bar{\mathcal{D}}_1-1)_1, (\bar{\mathcal{D}}_1-2), (\bar{\mathcal{D}}_1-11)_{-1}, (\bar{\mathcal{D}}_1-12)$;
- e) There is one simple left-symmetric algebra on \mathcal{D}_1 : $(\bar{\mathcal{D}}_1-10)$.

3.4 The left-symmetric algebras on \mathcal{D}_l , $0 < |l| < 1$ or $l = e^{i\theta}, 0 < \theta \leq \pi$

Throughout this subsection, without special saying, $0 < |l| < 1$ or $l = e^{i\theta}, 0 < \theta \leq \pi$. The automorphism group of \mathcal{D}_l is

$$\forall 0 < |l| < 1, \text{ or } l = e^{i\theta}, 0 < \theta < \pi, \text{ Aut}(\mathcal{D}_l) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \mid a_{11}a_{22} \neq 0 \right\}. \quad (3.7)$$

$$\text{Aut}(\mathcal{D}_{-1}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \mid a_{11}a_{22} \neq 0 \right\} \cup \left\{ \begin{pmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & -1 \end{pmatrix} \mid a_{12}a_{21} \neq 0 \right\}. \quad (3.8)$$

Proposition 3.12 The equivalent classes of the representations of \mathcal{D}_l are divided into the following cases:

$$\begin{aligned} \text{(AI)} \quad & f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(AII)} \quad & f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(AIII)} \quad & f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 1 & f_{11}(e_3) \end{pmatrix}. \\ \text{(AIV)} \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{22}(e_3) + l & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(AV)} \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) & 0 & 0 \\ 0 & f_{33}(e_3) - l & 0 \\ 1 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(AVI)} \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) & 0 & 0 \\ 0 & f_{33}(e_3) + l & 0 \\ 1 & 0 & f_{33}(e_3) \end{pmatrix}. \\ \text{(AVII)} \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 2l & 0 & 0 \\ 0 & f_{33}(e_3) + l & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \end{aligned}$$

$$\begin{aligned}
(\text{BI}) \quad f(e_1) &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + l & 0 & 0 \\ 0 & f_{33}(e_3) + l - 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{BII}) \quad f(e_1) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + l & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{BIII})_{\frac{1}{2}} \quad &\text{only for } l = \frac{1}{2},
\end{aligned}$$

$$f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 0 & f_{33}(e_3) + \frac{1}{2} & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}.$$

$$(\text{BIV}) \quad (l \neq -1) \quad f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{22}(e_3) + 1 & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}.$$

$$(\text{BV}) \quad (l \neq -1) \quad f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{33}(e_3) & 0 & 0 \\ 0 & f_{33}(e_3) - 1 & 0 \\ 1 & 0 & f_{33}(e_3) \end{pmatrix}.$$

$$(\text{BVI}) \quad (l \neq -1) \quad f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{33}(e_3) & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 1 & 0 & f_{33}(e_3) \end{pmatrix}.$$

$$(\text{CI}) \quad (l \neq -1) \quad f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 2 & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}.$$

Remark If for the latter four cases, we extend l to $l = -1$, then it is easy to see that the following cases of representations of \mathcal{D}_{-1} are equivalent:

$$(\text{AIV}) \sim (\text{BIV}), \quad (\text{AV}) \sim (\text{BV}), \quad (\text{AVI}) \sim (\text{BVI}), \quad (\text{AVII}) \sim (\text{CI}),$$

by the following linear isomorphism which is in $\text{Aut}(\mathcal{D}_{-1})$

$$e_1 \rightarrow e_2, \quad e_2 \rightarrow e_1, \quad e_3 \rightarrow -e_3.$$

Proposition 3.13 For a representation of \mathcal{D}_l given in the above cases respectively, there exist bijective 1-cocycles if and only if it is equivalent to one of the following corresponding cases:

Case (AI): $f_{11}(e_3) = 1, f_{22}(e_3) = l, f_{33}(e_3) = \lambda, \lambda \in \mathbf{C}$.

$$\forall \lambda \in \mathbf{C}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (D_l - 1)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & \lambda e_3 \end{pmatrix}, \lambda \in \mathbf{C}.$$

Case (AII): (AII-1) $f_{11}(e_3) = l, f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies (D_l - 2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & le_3 + e_2 \end{pmatrix}.$$

Case(AII-2) $f_{11}(e_3) = 1, f_{33}(e_3) = l$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow (D_l - 3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & e_3 + e_1 \end{pmatrix}.$$

Case (AIII): there does not exist any bijective 1-cocycle;

Case (AIV): (AIV-1) $f_{22}(e_3) = \lambda, f_{33}(e_3) = 1, \lambda \neq 0$.

$$\forall \lambda \neq 0, C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix} \Rightarrow (D_l - 4)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda e_2 \\ e_1 & (\lambda + l)e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -l$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 0 & 1 \\ l & 0 & 0 \\ 1 & -l & 0 \end{pmatrix} \Rightarrow (D_l - 5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -le_2 \\ e_1 & 0 & -le_3 + e_2 \end{pmatrix};$$

when $\lambda = l$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow (D_l - 6) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_3 & 0 \\ e_1 & le_2 & 2le_3 \end{pmatrix}.$$

(AIV-2): $f_{22}(e_3) = 1 - l, f_{33}(e_3) = l$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{1-l} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow (D_l - 7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_1 \\ e_1 & le_2 + e_1 & (1-l)e_3 \end{pmatrix}.$$

(AIV-3) $_{\frac{1}{2}}$: (only for $l = \frac{1}{2}$) $f_{22}(e_3) = \frac{1}{2}, f_{33}(e_3) = \lambda, \lambda \in \mathbf{C}$.

$$\forall \lambda \in \mathbf{C}, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow (D_{\frac{1}{2}} - S - 1)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 & \lambda e_3 \end{pmatrix}, \lambda \in \mathbf{C}.$$

(Notice that when $\lambda = \frac{1}{2}$, it need to add $(D_{\frac{1}{2}} - 7)$; when $\lambda = 1$, it need to add $(D_{\frac{1}{2}} - 4)_{\frac{1}{2}}$ and $(D_{\frac{1}{2}} - 6)$).

Case (AV): for $l \neq \frac{1}{2}$, there does not exist any bijective 1-cocycle; for $l = \frac{1}{2}$: $f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow (D_{\frac{1}{2}} - S - 2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 & e_3 + e_1 \end{pmatrix}.$$

Case (AVI): (AVI-1) $f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow (D_l - 8) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_2 \\ e_1 & (1+l)e_2 & e_3 + e_1 \end{pmatrix};$$

for $l = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \implies (D_{-1} - T - 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_2 \\ e_1 & 0 & e_3 + e_1 + e_2 \end{pmatrix}.$$

(AVI-2) $_{\frac{1}{2}}$: (only for $l = \frac{1}{2}$) $f_{33}(e_3) = \frac{1}{2}$.

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 1 \end{pmatrix} \implies (D_{\frac{1}{2}} - S - 3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 + e_1 & \frac{1}{2}e_3 + e_2 \end{pmatrix}.$$

Case (AVII): for $l = \frac{1}{2}$, there does not exist any bijective 1-cocycle; for $l \neq \frac{1}{2}$: $f_{33}(e_3) = 1 - 2l$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - 2l \end{pmatrix} \implies (D_{l \neq \frac{1}{2}} - S - 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & (1 - 2l)e_2 \\ e_1 & (1 - l)e_2 & (1 - 2l)e_3 \end{pmatrix};$$

for $l = \frac{1}{3}$, there is an additional equivalent class:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \implies (D_{\frac{1}{3}} - T - 1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_3 & e_1 \\ e_1 & \frac{1}{3}e_2 + e_1 & \frac{2}{3}e_3 \end{pmatrix}.$$

Case (BI): (BI-1) $f_{33}(e_3) = 2 - l$.

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 - l \end{pmatrix} \implies (D_l - 9) = \begin{pmatrix} e_2 & 0 & 0 \\ 0 & 0 & (2 - l)e_2 \\ e_1 & 2e_2 & (2 - l)e_3 \end{pmatrix}.$$

(BI-2) $f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \implies (D_l - 10) = \begin{pmatrix} 0 & e_3 & 0 \\ e_3 & 0 & 0 \\ e_1 & le_2 & (l + 1)e_3 \end{pmatrix}.$$

(BI-3) $_{l \neq \frac{1}{2}}$: (only for $l \neq \frac{1}{2}$) $f_{33}(e_3) = l$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2l - 1 & 0 \end{pmatrix} \implies (D_{l \neq \frac{1}{2}} - S - 2) = \begin{pmatrix} 0 & 0 & (2l - 1)e_1 \\ 0 & e_1 & 0 \\ 2le_1 & le_2 & (2l - 1) \end{pmatrix}.$$

Case (BII): (BII-1) $f_{33}(e_3) = \lambda, \lambda \neq 0$.

$$\forall \lambda \neq 0, C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \implies (D_l - 11)_\lambda = \begin{pmatrix} 0 & 0 & \lambda e_1 \\ 0 & 0 & \lambda e_2 \\ (\lambda + 1)e_1 & (\lambda + l)e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \implies (D_l - 12) = \begin{pmatrix} 0 & 0 & -e_1 \\ 0 & 0 & -e_2 \\ 0 & (l - 1)e_2 & -e_3 + e_1 \end{pmatrix};$$

when $\lambda = -l$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{l} & 0 & -l \end{pmatrix} \Rightarrow (D_l - 13) = \begin{pmatrix} 0 & 0 & -le_1 \\ 0 & 0 & -le_2 \\ (1-l)e_1 & 0 & -le_3 + e_2 \end{pmatrix};$$

when $\lambda = l - 1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & \frac{1}{1-l} & 0 \\ 0 & 0 & l-1 \end{pmatrix} \Rightarrow (D_l - 14) = \begin{pmatrix} 0 & 0 & (l-1)e_1 \\ 0 & 0 & (l-1)e_2 + e_1 \\ le_1 & (2l-1)e_2 + e_1 & (l-1)e_3 \end{pmatrix};$$

when $\lambda = 1 - l$, there is an additional equivalent class:

$$C = \begin{pmatrix} \frac{1}{l-1} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1-l \end{pmatrix} \Rightarrow (D_l - 15) = \begin{pmatrix} 0 & 0 & (1-l)e_1 + e_2 \\ 0 & 0 & (1-l)e_2 \\ (2-l)e_1 + e_2 & e_2 & (1-l)e_3 \end{pmatrix}.$$

(BII-2) $_{\frac{1}{2}}$: (only for $l = \frac{1}{2}$) $f_{33}(e_3) = -\frac{1}{2}$. Besides $(D_{\frac{1}{2}} - 11)_{-\frac{1}{2}}$, $(D_{\frac{1}{2}} - 13)$ and $(D_{\frac{1}{2}} - 14)$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & -\frac{1}{2} \end{pmatrix} \Rightarrow (D_{\frac{1}{2}} - S - 4) = \begin{pmatrix} 0 & 0 & -\frac{1}{2}e_1 \\ 0 & 0 & -\frac{1}{2}e_2 + e_1 \\ \frac{1}{2}e_1 & e_1 & -\frac{1}{2}e_3 + e_2 \end{pmatrix}.$$

Case (BIII) $_{\frac{1}{2}}$: $f_{33}(e_3) = \lambda, \lambda \neq 0$.

$$\forall \lambda \neq 0, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow (D_{\frac{1}{2}} - S - 5)_{\lambda} = \begin{pmatrix} 0 & 0 & \lambda e_1 \\ 0 & e_1 & \lambda e_2 \\ (\lambda+1)e_1 & (\lambda+\frac{1}{2})e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \Rightarrow (D_{\frac{1}{2}} - S - 6) = \begin{pmatrix} 0 & 0 & -e_1 \\ 0 & e_1 & -e_2 \\ 0 & -\frac{1}{2}e_2 & -e_3 + e_1 \end{pmatrix}.$$

when $\lambda = \frac{1}{2}$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow (D_{\frac{1}{2}} - S - 7) = \begin{pmatrix} 0 & e_3 & 0 \\ e_3 & e_1 & 0 \\ e_1 & \frac{1}{2}e_2 & \frac{3}{2}e_3 \end{pmatrix}.$$

Case (BIV): (BIV-1) ($l \neq -1$) $f_{22}(e_3) = \lambda, f_{33}(e_3) = l, \lambda \neq 0$.

$$\forall \lambda \neq 0, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \lambda & 0 \end{pmatrix} \Rightarrow (D_{l \neq -1} - 16)_{\lambda} = \begin{pmatrix} 0 & 0 & \lambda e_1 \\ 0 & 0 & 0 \\ (\lambda+1)e_1 & le_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \Rightarrow (D_{l \neq -1} - 17) = \begin{pmatrix} 0 & 0 & -e_1 \\ 0 & 0 & 0 \\ 0 & le_2 & -e_3 + e_1 \end{pmatrix}.$$

when $\lambda = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \implies (D_{l \neq -1} - 18) = \begin{pmatrix} e_3 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & le_2 & 2e_3 \end{pmatrix}.$$

$$(BIV-2) \quad (l \neq -1) \quad f_{22}(e_3) = l - 1, f_{33}(e_3) = 1.$$

$$C = \begin{pmatrix} \frac{1}{l-1} & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \implies (D_{l \neq -1} - 19) = \begin{pmatrix} 0 & 0 & e_2 \\ 0 & 0 & 0 \\ e_1 + e_2 & le_2 & (l-1)e_3 \end{pmatrix}.$$

Case (BV): $(l \neq -1)$ there does not exist any bijective 1-cocycle;

Case (BVI): $(l \neq -1) \quad f_{33}(e_3) = l.$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ l^2 & 0 & 0 \\ 0 & 0 & l \end{pmatrix} \implies (D_{l \neq -1} - 20) = \begin{pmatrix} 0 & 0 & le_1 \\ 0 & 0 & 0 \\ (1+l)e_1 & le_2 & le_3 + e_2 \end{pmatrix}.$$

Case (CI): $(l \neq -1) \quad f_{33}(e_3) = l - 2.$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & l-2 \end{pmatrix} \implies (D_{l \neq -1} - 21) = \begin{pmatrix} e_2 & 0 & (l-2)e_1 \\ 0 & 0 & 0 \\ (l-1)e_1 & le_2 & (l-2)e_3 \end{pmatrix}.$$

Remark 1 It is easy to see that we can extend the extent of some parameters appearing in above left-symmetric algebras:

$$(D_l - 1)_{\lambda=0} \cong (D_l - 4)_{\lambda=0} \cong (D_l - 11)_{\lambda=0}; \quad (D_{\frac{1}{2}} - S - 1)_{\lambda=0} \cong (D_{\frac{1}{2}} - S - 5)_{\lambda=0}.$$

Remark 2 In some sense, we can extend the value of l to $l = 1$ to get the classification of left-symmetric algebras on \mathcal{D}_1 :

$$\begin{aligned} (\bar{D} - 1)_{\lambda} &\cong (D_{l=1} - 1)_{\lambda}; \quad (\bar{D} - 2) \cong (D_{l=1} - 2)_{\lambda} \cong (D_{l=1} - 3); \\ (\bar{D} - 3)_{\lambda} &\cong (D_{l=1} - 4)_{\lambda} \cong (D_{l=1} - 16)_{\lambda}; \quad (\bar{D} - 4) \cong (D_{l=1} - 5) \cong (D_{l=1} - 17); \\ (\bar{D} - 6) &\cong (D_{l=1} - 7) \cong (D_{l=1} - 14) \cong (D_{l=1} - 15) \cong (D_{l=1} - 19); \\ (\bar{D} - 5) &\cong (D_{l=1} - 6) \cong (D_{l=1} - 18); \quad (\bar{D} - 7) \cong (D_{l=1} - 8) \cong (D_{l=1} - 20); \\ (\bar{D} - 8) &\cong (D_{l=1} - S - 1) \cong (D_{l=1} - 21); \quad (\bar{D} - 9) \cong (D_{l=1} - 9) \cong (D_{l=1} - S - 2); \\ (\bar{D} - 10) &\cong (D_{l=1} - 10); \quad (\bar{D} - 11)_{\lambda} \cong (D_{l=1} - 11)_{\lambda}; \quad (\bar{D} - 12) \cong (D_{l=1} - 12) \cong (D_{l=1} - 13). \end{aligned}$$

However, for some cases, the corresponding bijective is quite different. For example, $(\bar{D} - 6)$ belongs to the case (AV) of \mathcal{D}_1 , but $(D_l - 7)$ belongs to the case (AIV).

Remark 3 Similarly, we can extend the value of l to $l = 0$ to get certain left-symmetric algebras on \mathcal{N} :

$$\begin{aligned}
& (N-1)_\lambda \cong (D_{l=0}-1)_\lambda; \quad (N-5) \cong (D_{l=0}-2) \cong (D_{l=0}-13) \cong (D_{l=0}-20); \\
& (N-6) \cong (D_{l=0}-3); \quad (N-9)_\lambda \cong (D_{l=0}-4)_\lambda; \quad (N-5) \cong (D_{l=0}-5); \quad (N-10) \cong (D_{l=0}-6); \\
& (N-11) \cong (D_{l=0}-7); \quad (N-15) \cong (D_{l=0}-8); \quad (N-16) \cong (D_{l=0}-S-1); \\
& (N-36) \cong (D_{l=0}-9); \quad (N-35) \cong (D_{l=0}-10); \quad (N-33) \cong (D_{l=0}-S-2); \\
& (N-38)_\lambda \cong (D_{l=0}-11)_\lambda; \quad (N-39) \cong (D_{l=0}-12); \quad (N-40) \cong (D_{l=0}-14); \\
& (N-42) \cong (D_{l=0}-15); \quad (N-18)_\lambda \cong (D_{l=0}-16)_\lambda; \quad (N-19) \cong (D_{l=0}-17); \\
& (N-20) \cong (D_{l=0}-18); \quad (N-21) \cong (D_{l=0}-19); \quad (N-44) \cong (D_{l=0}-21).
\end{aligned}$$

All above algebras satisfy the condition: $f_{11}(e_1) = f_{22}(e_1) = f_{33}(e_1) = 0$. However, there are certain left-symmetric algebras on \mathcal{N} satisfying this condition such as (N-12), (N-17), (N-34) and (N-41) which cannot be obtained from \mathcal{D}_l as $l = 0$.

Proposition 3.14 With the notations as above, among the left-symmetric algebras on \mathcal{D}_l , we have

- a) Associative algebras: $(D_{-1}-4)_1$;
- b) Transitive left-symmetric algebras: $(D_l-1)_0$; $(D_{-1}-10)$; $(D_{\frac{1}{2}}-S-1)_0$;
- c) Novikov algebras: $(D_l-1)_0$; $(D_l-11)_\lambda$ ($\lambda \neq 0$); (D_1-12) ; (D_1-13) ; $(D_{\frac{1}{2}}-14)$; $(D_{\frac{1}{2}}-S-1)_0$; $(D_{\frac{1}{2}}-S-4)$; $(D_{\frac{1}{2}}-S-5)_\lambda$ ($\lambda \neq 0$); $(D_{\frac{1}{2}}-S-6)$;
- d) Bi-symmetric algebras: $(D_{-1}-4)_1$; $(D_{-1}-5)$; $(D_{-1}-8)$; $(D_{-1}-T-1)$;
- e) Simple left-symmetric algebras on: (D_l-10) ; $(D_{\frac{1}{2}}-S-7)$.

3.5 The left-symmetric algebras on \mathcal{E}

The automorphism group of \mathcal{E} is

$$\text{Aut}(\mathcal{E}) = \left\{ \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \mid a_{11}^2 \neq 0 \right\}. \quad (3.8)$$

Proposition 3.15 The equivalent classes of the representations of \mathcal{E} are divided into the following cases :

$$(\text{AI}) \quad f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}.$$

$$\begin{aligned}
(\text{AII}) \quad & f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{AIII}) \quad & f(e_1) = 0, f(e_2) = 0, f(e_3) = \begin{pmatrix} f_{11}(e_3) & 0 & 0 \\ 1 & f_{11}(e_3) & 0 \\ 0 & 1 & f_{11}(e_3) \end{pmatrix}. \\
(\text{AIV}) \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{22}(e_3) + 1 & 0 & 0 \\ 0 & f_{22}(e_3) & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{AV}) \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) & 0 & 0 \\ 0 & f_{33}(e_3) - 1 & 0 \\ 1 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{AVI}) \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 1 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{AVII}) \quad & f(e_1) = 0, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 2 & 0 & 0 \\ 0 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{BI}) \quad & f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 1 & f_{33}(e_3) + 1 & 0 \\ 0 & 0 & f_{33}(e_3) \end{pmatrix}. \\
(\text{BII}) \quad & f(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, f(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, f(e_3) = \begin{pmatrix} f_{33}(e_3) + 1 & 0 & 0 \\ 0 & f_{33}(e_3) & 0 \\ 0 & -1 & f_{33}(e_3) \end{pmatrix}.
\end{aligned}$$

Proposition 3.16 For a representation of \mathcal{E} given in the above cases respectively, there exist

bijjective 1-cocycles if and only if it is equivalent to one of the following corresponding cases:

Case (AI): there does not exist any bijjective 1-cocycle;

Case (AII): $f_{11}(e_3) = 1, f_{33}(e_3) = \lambda, \lambda \in \mathbf{C}$.

$$\forall \lambda \in \mathbf{C}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (E - 1)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & \lambda e_3 \end{pmatrix}, \lambda \in \mathbf{C}.$$

Case (AIII): $f_{11}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \implies (E - 2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_1 & e_1 + e_2 & e_3 + e_2 \end{pmatrix}.$$

Case (AIV): $f_{22}(e_3) = 0, f_{33}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \implies (E - 3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -e_1 \\ e_1 & e_2 & 0 \end{pmatrix}.$$

Case (AV): $f_{33}(e_3) = 1$.

$$C_\lambda = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{\lambda+1}{\lambda} \\ 0 & 1 & 0 \end{pmatrix}, \lambda \neq -1, 0 \implies (E - 4)_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda e_1 \\ e_1 & e_2 + (\lambda + 1)e_1 & 0 \end{pmatrix}, \lambda \neq 0, -1.$$

Case (AVI): $f_{11}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow (E - 5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_2 - e_3 & e_2 - e_3 \\ e_1 & e_1 + 2e_2 - e_3 & e_3 + e_1 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_3 & 0 \\ e_1 & e_1 + e_2 & 2e_3 \end{pmatrix}.$$

Case (AVII): $f_{33}(e_3) = -1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & -1 \end{pmatrix} \Rightarrow (E - 6) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_1 & -e_1 - e_2 \\ e_1 & 0 & -e_3 - e_2 \end{pmatrix}.$$

Case (BI): $f_{33}(e_3) = \lambda, \lambda \neq 0$.

$$\forall \lambda \neq 0, C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \Rightarrow (E - 7)_\lambda = \begin{pmatrix} 0 & 0 & \lambda e_1 \\ 0 & 0 & \lambda e_2 \\ (\lambda + 1)e_1 & e_1 + (\lambda + 1)e_2 & \lambda e_3 \end{pmatrix}, \lambda \neq 0;$$

when $\lambda = -1$, there is an additional equivalent class:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix} \Rightarrow (E - 8) = \begin{pmatrix} 0 & 0 & -e_1 \\ 0 & 0 & -e_2 \\ 0 & e_1 & -e_3 + e_2 \end{pmatrix}.$$

Case (BII): $f_{11}(e_3) = 1$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \Rightarrow (E - 9) = \begin{pmatrix} 0 & 0 & e_1 \\ 0 & e_1 & 0 \\ 2e_1 & e_1 + e_2 & e_3 - e_2 \end{pmatrix}.$$

Remark It is easy to see that we can extend the extent of some parameters appearing in above left-symmetric algebras:

$$(E - 1)_{\lambda=0} \cong (E - 4)_{\lambda=0} \cong (E - 7)_{\lambda=0}; \quad (E - 4)_{\lambda=-1} \cong (E - 3).$$

Proposition 3.17 With the notations as above. Among the left-symmetric algebras on \mathcal{E} , we have

- a) There is not any associative algebra on \mathcal{E} ;
- b) Transitive left-symmetric algebras: $(E-1)_0; (E-3); (E-4)_\lambda (\lambda \neq 0, -1)$;
- c) Novikov algebras: $(E-1)_0; (E-7)_\lambda (\lambda \neq 0); (E-8)$;
- d) There is not any bi-symmetric algebra on \mathcal{E} ;
- e) There is not any simple left-symmetric algebra on \mathcal{E} .

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References

- [AS] A. Andrada, S. Salamon, Complex product structure on Lie algebras, *Forum Math.* 17 (2005) 261-295.
- [Bai] C.M. Bai, A unified algebraic approach to the classical Yang-Baxter equation, *J. Phys. A: Math. Theor.* 40 (2007) 11073-11082.
- [BM1] C.M. Bai, D.J. Meng, The classification of left-symmetric algebra in dimension 2, (in Chinese), *Chinese Science Bulletin* 23 (1996) 2207.
- [BM2] C.M. Bai, D.J. Meng, The structure of bi-symmetric algebras and their sub-adjacent Lie algebras, *Comm. in Algebra* 28 (2000) 2717-2734.
- [BM3] C.M. Bai, D.J. Meng, The classification of Novikov algebras in low dimensions, *J. Phys. A: Math. Gen.* 34 (2001) 1581-1594.
- [BM4] C.M. Bai, D.J. Meng, On the realization of transitive Novikov algebras, *J. Phys. A: Math. Gen.* 34 (2001) 3363-3372.
- [BM5] C.M. Bai, D.J. Meng, The realizations of non-transitive Novikov algebras, *J. Phys. A: Math. Gen.* 34 (2001) 6435-6442.
- [BM6] C.M. Bai, D.J. Meng, A Lie algebraic approach to Novikov algebras, *J. Geo. Phys.* 45 (2003) 218-230.
- [BK] B. Bakalov, V. Kac, Field algebras, *Int. Math. Res. Not.* (2003) 123-159.
- [BN] A.A. Balinskii, S.P. Novikov, Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras, *Soviet Math. Dokl.* 32 (1985) 228-231.

- [Bar] M.L. Barbeis, Hypercomplex structures on four-dimensional Lie groups, Proc. Amer. Math. Soc. 125 (1997) 1043-1054.
- [Bau] O. Baues, Left-symmetric algebras for $gl(n)$, Trans. Amer. Math. Soc. 351 (1999) 2979-2996.
- [Bo] M. Bordemann, Generalized Lax pairs, the modified classical Yang-Baxter equation, and affine geometry of Lie groups, Comm. Math. Phys. 135 (1990) 201-216.
- [Bu1] D. Burde, Left-invariant affine structures on reductive Lie groups, J. Algebra 181 (1996) 884-902.
- [Bu2] D. Burde, Simple left-symmetric algebras with solvable Lie algebra, Manuscripta Math. 95 (1998) 397-411.
- [Bu2] D. Burde, Left-symmetric algebras, or pre-Lie algebras in geometry and physics, Cent. Eur. J. Math. 4 (2006) 323-357.
- [C] A. Cayley, On the theory of analytic forms called trees, Collected Mathematical Papers of Arthur Cayley, Cambridge Univ. Press, Vol. 3 (1890) 242-246.
- [CL] F. Chapoton, M. Livernet, Pre-Lie algebras and the rooted trees operad, Int. Math. Res. Not. (2001) 395-408.
- [CK] A. Connes, D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Comm. Math. Phys. 199 (1998) 203-242.
- [DM] A. Diatta, A. Medina, Classical Yang-Baxter equation and left-invariant affine geometry on Lie groups, arXiv:math.DG/0203198.
- [DN] B.A. Dubrovin, S.P. Novikov, On Poisson brackets of hydrodynamic type, Soviet Math. Dokl. 30 (1984) 651-654.
- [ESS] P. Etingof, T. Schedler, A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equations, Duke Math. J. 100 (1999) 169-209.
- [F] V.T. Filipov, A class of simple nonassociative algebras, Mat. Zametki 45 (1989) 101-105.
- [GD] I.M. Gel'fand, I. Ya. Dorfman, Hamiltonian operators and algebraic structures related to them, Funct. Anal. Appl. 13 (1979) 248-262.
- [G] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. Math. 78 (1963) 267-288.
- [GS] I.Z. Golubschik, V.V. Sokolov, Generalized operator Yang-Baxter equations, integrable ODES and nonassociative algebras, J. Nonlinear Math. Phys., 7 (2000) 184-197.
- [H] Helmstetter, J. Radical d'une algebre symetrique a gauche. Ann. Inst. Fourier (Grenoble) 29 (1979) 17-35.

- [J] N. Jacobson, Lie algebras, Inertscience, New York (1962).
- [Ki1] H. Kim, Complete left-invariant affine structures on nilpotent Lie groups, J. Differential Geometry 24 (1986) 373-394.
- [Ki2] H. Kim, Extensions of left-symmetric algebras, Algebras, Groups and Geometries 4 (1987) 73-117.
- [Kl] E. Kleinfeld, Assosymmetric rings, Proc. Amer. Math. Soc. 8 (1957) 983-986.
- [Ko] J.-L. Koszul, Domaines bornés homogènes et orbites de groupes de transformations affines, Bull. Soc. Math. France 89 (1961) 515-533.
- [Ku1] B.A. Kupershmidt, Non-abelian phase spaces, J. Phys. A: Math. Gen. 27 (1994) 2801-2810.
- [Ku2] B.A. Kupershmidt, On the nature of the Virasoro algebra, J. Nonlinear Math. Phys. 6 (1999) 222-245.
- [Ku3] B.A. Kupershmidt, What a classical r -matrix really is, J. Nonlinear Math. Phys. 6 (1999) 448-488.
- [LM] A. Lichnerowicz, A. Medina, On Lie groups with left invariant symplectic or kahlerian structures, Lett. Math. Phys. 16 (1988) 225-235.
- [Ma] Y. Matsushima, Affine structures on complex mainfolds, Osaka J. Math. 5 (1968) 215-222.
- [Me] A. Medina, Flat left-invariant connections adapted to the automorphism structure of a Lie group, J. Differential Geometry 16 (1981) 445-474.
- [O] J.M. Osborn, Novikov algebras, Nova J. Algebra Geom. 1 (1992) 1-14.
- [SW] A.A. Sagle, R.E. Walde, Introduction to Lie groups and Lie algebras, Academic Press, New York (1973).
- [S] M.A. Semonov-Tian-Shansky, What is a classical R-matrix? Funct. Anal. Appl. 17 (1983) 259-272.
- [V] E.B. Vinberg, Convex homogeneous cones, Transl. of Moscow Math. Soc. No. 12 (1963) 340-403.
- [X] X. Xu, On simple Novikov algebras and their irreducible modules, J. Algebra 185 (1996) 905-934.
- [Z] E.I. Zel'manov, On a class of local translation invariant Lie algebras, Soviet Math. Dokl. 35 (1987) 216-218.